

GENUS ONE FACTORS OF CURVES DEFINED BY SEPARATED VARIABLE POLYNOMIALS

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ABSTRACT. We give some sufficient conditions on complex polynomials P and Q to assure that the algebraic plane curve $P(x) - Q(y) = 0$ has no irreducible component of genus 0 or 1. Moreover, if $\deg(P) = \deg(Q)$ and if both P, Q satisfy Hypothesis I introduced by H. Fujimoto, our sufficient conditions are necessary.

1. INTRODUCTION

Give two polynomials P and Q in one variable over a field \mathbf{K} of characteristic $p \geq 0$, two questions naturally arise: First, function theorists have found it interesting to ask when does the equation $P(f) = Q(g)$ have a nontrivial functional solution (f, g) ? Second, number theorists want to know whether there are finitely many or infinitely many \mathbf{K} -rational solutions to the equation $P(x) = Q(y)$ when \mathbf{K} is a number field, or possibly a global field of positive characteristic. The two questions are related in certain cases by theorems of Faltings and Picard: When $\mathbf{K} = \mathbb{C}$, Picard's theorem says $P(f) = Q(g)$ has no solutions consisting of non-constant meromorphic functions f and g when the plane curve $P(x) = Q(y)$ has no irreducible components of (geometric) genus 0 or 1. Similarly, Faltings's Theorem says that if the plane curve $P(x) = Q(y)$ has no irreducible components of (geometric) genus less than two, then for each number field \mathbf{K} over which P and Q are defined, there are only finitely many \mathbf{K} -rational solutions to $P(x) = Q(y)$.

When the degrees of P and Q are relative prime, one knows by Ehrenfeucht's criterion ([12], [19]) that the plane curve $P(x) = Q(y)$ is irreducible. In this case, Ritt's second theorem completely characterizes when the curve has genus zero (see [18, pp 40-41]), and Avanzi and Zannier in [6] completely characterize the case of genus one. In [13], Fried gave conditions such that the curve has genus zero when $\gcd(\deg P, \deg Q) \leq 2$ and also for arbitrary $d = \gcd(\deg P, \deg Q)$ provided the

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degrees of P and Q are larger than some number $N(d)$ and the curve is irreducible. Most of results of this type suppose the irreducibility of the curve, however, when $\gcd(\deg P, \deg Q) > 1$, the problem of determining the irreducibility of $P(x) - Q(y)$ remains wide open.

We consider now the case of the complex field \mathbb{C} . In some special cases and under the assumption that P is indecomposable (that is, P cannot be written as a composition of two polynomials of degree larger than 1), Tverberg determined in [19, Ch. 2] whether $[P(x) - P(y)]/(x - y)$ could contain a linear or quadratic factor. Similarly, Bilu [8] determined all the pairs of polynomials such that $P(x) - Q(y)$ contains a quadratic factor. Avanzi and Zannier in [5] give a nice characterization of when a curve of the form $P(x) = cP(y)$ has genus at least 1, where c is a nonzero complex constant. In the case of polynomials satisfying Fujimoto's hypothesis I, (i.e when restricted to the zero set of its derivative P' , the polynomial P is injective), complete characterizations for when all the irreducible components of curve $P(x) - Q(y) = 0$ have genus at least 2 have been given in [2], [4], [10], [15] and also [14].

In this paper, we will give some sufficient conditions that the plane curve $P(x) = Q(y)$ has no irreducible component of (geometric) genus 0 or 1 for complex polynomials P and Q , not necessarily satisfying Fujimoto's hypothesis I.

Henceforth, all polynomials belong to $\mathbb{C}[X]$ and all curves we consider are defined in $\mathbb{P}^2(\mathbb{C})$. We denote the coefficients of P and Q by

$$\begin{aligned} P(X) &= a_0 + a_1X + \dots + a_{n_0-1}X^{n_0-1} + a_{n_0}X^{n_0} + a_nX^n, \\ (1) \quad Q(X) &= b_0 + b_1X + \dots + b_{m_0-1}X^{m_0-1} + b_{m_0}X^{m_0} + b_mX^m, \end{aligned}$$

where a_n, a_{n_0}, b_{m_0} and b_m are non-zero.

Without loss of generality, throughout the paper we will assume that $n \geq m$.

If one of the polynomials P or Q is linear, say $P(x) = ax + b$, then $(\frac{1}{a}Q(f) - b, f)$ is a solution of the equation $P(x) = Q(y)$, where f is any non-constant meromorphic function. Hence, from now on, we always assume that both P and Q are not linear polynomials.

The first result is:

Theorem 1. *Let $m = n$ and $n \geq \max\{n_0, m_0\} + 4$. Suppose that $P(x) - Q(y)$ has no linear factor. Then the plane curve $P(x) = Q(y)$ has no irreducible component of genus 0 or 1.*

We will denote by $\alpha_1, \alpha_2, \dots, \alpha_l$ and $\beta_1, \beta_2, \dots, \beta_h$ the distinct roots of $P'(X)$ and $Q'(X)$, respectively. We will use p_1, p_2, \dots, p_l and q_1, q_2, \dots, q_h to denote the

multiplicities of the roots in $P'(X)$ and $Q'(X)$, respectively. Thus,

$$\begin{aligned} P'(X) &= na_n(X - \alpha_1)^{p_1}(X - \alpha_2)^{p_2} \dots (X - \alpha_l)^{p_l}, \\ Q'(X) &= mb_m(X - \beta_1)^{q_1}(X - \beta_2)^{q_2} \dots (X - \beta_h)^{q_h}. \end{aligned}$$

The polynomial $P(X)$ is said to satisfy *Hypothesis I* if

$$P(\alpha_i) \neq P(\alpha_j) \text{ whenever } i \neq j, \ i, j = 1, 2, \dots, l,$$

or in other words P is injective on the roots of P' .

In order to state the theorems clearly, we need to introduce the following notation:

Notation. We put:

$$\begin{aligned} A_0 &:= \{(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq h, P(\alpha_i) = Q(\beta_j)\}, \\ A_1 &:= \{(i, j) \mid (i, j) \in A_0, p_i > q_j\}, \\ A_2 &:= \{(i, j) \mid (i, j) \in A_0, p_i < q_j\}. \end{aligned}$$

and we put $l_0 := \#A_0$.

The main results are as follows.

Theorem 2. *Let $P(X)$ and $Q(X)$ satisfy Hypothesis I and suppose $P(x) - Q(y)$ has no linear factor. Then, if*

$$\sum_{(i,j) \in A_1} (p_i - q_j) + \sum_{1 \leq i \leq l, (i,j) \notin A_0} p_i \geq n - m + 3,$$

then the curve $P(x) - Q(y) = 0$ has no irreducible component of genus 0 or 1.

Corollary 3. *With the same conditions as in Theorem 2, then $P(x) - Q(y)$ has no factor of genus 0 or 1 if the following holds*

$$\sum_{(i,j) \in A_2} (q_j - p_i) + \sum_{1 \leq j \leq h, (i,j) \notin A_0} q_j \geq 3.$$

When both the polynomials P and Q satisfy Hypothesis I and their degrees are the same, we are able to give a sufficient and necessary condition to assure that the curve has no irreducible components of genus 0 or 1.

Theorem 4. *Let P and Q be polynomials satisfying Hypothesis I and $\deg P = \deg Q$. Then the curve $P(x) - Q(y)$ has no factor of genus 0 or 1 if and only if after possibly changing indices none of the following hold:*

- (1) $P(x) - Q(y)$ has a linear factor.
- (2) $n = 2$ or $n = 3$.
- (3) $n = 4$ and either there exists at least two i such that $P(\alpha_i) = Q(\beta_i)$ or there exists only one i such that $P(\alpha_i) = Q(\beta_i)$ and $|p_i - q_i| = 2$.

- (4) either $n = p_1 + 1$, $l = 1$, $h = 2$, $p_1 = q_1 + 1$, $q_2 = 1$ and $P(\alpha_1) = Q(\beta_1)$; or
 $n = p_1 + 2$, $h = 1$, $l = 2$, $q_1 = p_1 + 1$, $p_2 = 1$ and $P(\alpha_1) = Q(\beta_1)$.
- (5) $l = h = 2$, $p_2 = q_2 = 1$, $p_1 = q_1$, $n = p_1 + 2$, and $P(\alpha_1) = Q(\beta_1)$.
- (6) $n = 5$, $l_0 = l = h = 3$, $p_3 = p_2 = q_2 = q_3 = 1$, $p_1 = q_1 = 2$, $P(\alpha_i) = Q(\beta_i)$,
for $i = 1, 2, 3$.
- (7) $n = 5$, $l_0 = l = h = 2$, $p_i = q_i = 2$, $P(\alpha_i) = Q(\beta_i)$, for $i = 1, 2$.

A main technique to prove these results is constructing two non-trivial regular 1-forms. This method helps to avoid a difficulty of proving irreducibility of the curve.

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2. A KEY LEMMA

We first recall some notation (for more detail, see [4, Section 2]).

Let $F(z_0, z_1, z_2)$ be a homogeneous polynomial of degree n and let

$$C = \{[z_0, z_1, z_2] \in \mathbb{P}^2(\mathbb{C}) \mid F(z_0, z_1, z_2) = 0\}.$$

By Euler's theorem, for $[z_0, z_1, z_2] \in C$, we have

$$(2) \quad z_0 \frac{\partial F}{\partial z_0} + z_1 \frac{\partial F}{\partial z_1} + z_2 \frac{\partial F}{\partial z_2} = 0.$$

The equation of the tangent space of C at the point $[z_0, z_1, z_2] \in C$ is defined by

$$(3) \quad dx \frac{\partial F}{\partial z_0} + dy \frac{\partial F}{\partial z_1} + dz \frac{\partial F}{\partial z_2} = 0.$$

Then by Cramer's rule, on the curve C we have

$$\frac{\partial F}{\partial z_0} = \frac{W(z_1, z_2)}{W(z_0, z_1)} \frac{\partial F}{\partial z_2}, \quad \frac{\partial F}{\partial z_1} = \frac{W(z_2, z_0)}{W(z_0, z_1)} \frac{\partial F}{\partial z_2},$$

where $W(z_i, z_j)$ denotes the Wronskian of z_i and z_j , as

$$W(z_i, z_j) := \begin{vmatrix} z_i & z_j \\ dz_i & dz_j \end{vmatrix}.$$

Therefore

$$(4) \quad \frac{W(z_1, z_2)}{\frac{\partial F}{\partial z_0}} = \frac{W(z_2, z_0)}{\frac{\partial F}{\partial z_1}} = \frac{W(z_0, z_1)}{\frac{\partial F}{\partial z_2}}.$$

Definition 5. Let $C \subset \mathbb{P}^2(\mathbb{C})$ be an algebraic curve. A 1-form ω on C is said to be *regular* if it is the restriction (more precisely, the pull-back) of a rational 1-form on $\mathbb{P}^2(\mathbb{C})$ such that the pole set of ω does not intersect C . A 1-form is said to be of

Wronskian type if it is of the form $\frac{R}{S}W(z_i, z_j)$ for some homogeneous polynomials R and S such that $\deg S = \deg R + 2$.

Note that the condition in the above definition ensures a well-defined rational 1-form on $\mathbb{P}^2(\mathbb{C})$ since

$$\frac{R}{S}W(z_i, z_j) = \frac{z_j^2 R}{S} \frac{W(z_i, z_j)}{z_j^2}.$$

A holomorphic map

$$\phi = (\phi_0, \phi_1, \phi_2) : \Delta_\epsilon = \{t \in \mathbb{C} \mid |t| < \epsilon\} \rightarrow C, \quad \phi(0) = \mathfrak{p}$$

is referred to as a *holomorphic parameterization of C at \mathfrak{p}* . A local holomorphic parameterization exists for sufficiently small ϵ . A rational function Q on the curve C is represented by A/B where A and B are homogeneous polynomials in z_0, z_1, z_2 such that $B|_C$ is not identically zero. Thus $Q \circ \phi$ is a well-defined meromorphic function on Δ_ϵ with Laurent expansion

$$Q \circ \phi(t) = \sum_{i=m}^{\infty} a_i t^i, \quad a_m \neq 0.$$

The order of $Q \circ \phi$ at $t = 0$ is by definition m and shall be denoted by

$$(5) \quad \text{ord}_{\mathfrak{p}, \phi} Q = \text{ord}_{t=0} Q(\phi(t)).$$

The function $Q \circ \phi$ is holomorphic if and only if $m \geq 0$. The rational function Q is regular at \mathfrak{p} if and only if $Q \circ \phi$ is holomorphic for all local holomorphic parameterizations of C at \mathfrak{p} . From now, we write $\text{ord}_{\mathfrak{p}} Q$ instead of $\text{ord}_{\mathfrak{p}, \phi} Q$ for some holomorphic parameterization of C .

Lemma 6 (Key Lemma). *Let C be a projective curve of degree n in $\mathbb{P}^2(\mathbb{C})$ defined by $F(z_0, z_1, z_2) = 0$. Assume that there is $i \neq j \neq k \in \{0, 1, 2\}$ and two well-defined rational 1-forms of Wronskian type*

$$\omega_1 = \frac{R_1}{S_1} W(z_i, z_j), \quad \text{and } \omega_2 = \frac{R_2}{S_2} W(z_i, z_j)$$

which satisfy the following

- (i) S_1, S_2 are factors of $\frac{\partial F}{\partial z_k}$.
- (ii) ω_1 and ω_2 are \mathbb{C} -linearly independent on any irreducible component of the curve C .
- (iii) For $i = 1, 2$, ω_i is regular at every $\mathfrak{p} \in \mathcal{S} \cap S_i$, where \mathcal{S} is the set of singular points of C and S_i is the zero set of S_i ,

Then every irreducible component of the curve has genus at least 2.

Proof. The rational 1-form ω_1 has possible poles at $(z_0, z_1, z_2) \in \mathbb{P}^2$ such that $S_1(z_0, z_1, z_2) = 0$. By the hypothesis that S_1 is a factor of $\frac{\partial F}{\partial z_k}$, we can write

$$\frac{\partial F}{\partial z_k} = S_1 H_1,$$

which implies

$$\omega_1 = \frac{R_1 H_1 W(z_i, z_j)}{S_1 H_1} = \frac{R_1 H_1 W(z_i, z_j)}{\frac{\partial F}{\partial z_k}}.$$

Together with (4), we have

$$\omega_1 = \frac{R_1 H_1 W(z_1, z_2)}{\frac{\partial F}{\partial z_0}} = \frac{R_1 H_1 W(z_2, z_0)}{\frac{\partial F}{\partial z_1}} = \frac{R_1 H_1 W(z_0, z_1)}{\frac{\partial F}{\partial z_2}}.$$

Hence, ω_1 only has a possible pole at $(z_0, z_1, z_2) \in \mathcal{S} \cap \mathcal{S}_i$, which is impossible by the condition (iii). Therefore, ω_1 is regular on the curve C .

Similarly, ω_2 is regular on the curve C .

Together with the condition (ii), on the curve C , there are two regular 1-forms which are independent each irreducible component. So, they have genus at least 2. \square

Remark 7. Now, let $F(z_0, z_1, z_2)$ be the homogeneous polynomial of degree n obtained by homogenizing $P(x) - Q(y)$, and let C be the curve defined by $F(z_0, z_1, z_2) = 0$ in \mathbb{P}^2 . Obviously, the equation $P(x) = Q(y)$ has no non-constant meromorphic solution if and only if the curve C is Brody hyperbolic, meaning there are no non-constant holomorphic maps from \mathbb{C} into C . By Picard's theorem, this is equivalent to every irreducible component of the curve having genus at least 2. Therefore, if the Key lemma holds, then the equation $P(x) = Q(y)$ has no non-constant meromorphic functions solutions.

3. PROOF OF THEOREMS 1-4

Recall

$$\begin{aligned} P(x) &= a_0 + a_1 x + \dots + a_{n_0-1} x^{n_0-1} + a_{n_0} x^{n_0} + a_n x^n, \\ Q(x) &= b_0 + b_1 x + \dots + b_{m_0-1} x^{m_0-1} + b_{m_0} x^{m_0} + b_m x^m, \end{aligned}$$

where a_n, a_{n_0}, b_{m_0} and b_m are non-zero and their derivatives are expressed in the forms

$$\begin{aligned} P'(x) &= n a_n (x - \alpha_1)^{p_1} (x - \alpha_2)^{p_2} \dots (x - \alpha_l)^{p_l}, \\ Q'(x) &= m b_m (x - \beta_1)^{q_1} (x - \beta_2)^{q_2} \dots (x - \beta_h)^{q_h}. \end{aligned}$$

Recall also that we are assuming $n \geq m$.

As in the remark at the end of the last section, let $F(z_0, z_1, z_2)$ be the homogeneous polynomial of degree n obtained by homogenizing $P(x) - Q(y)$, and let C be the curve defined by $F(z_0, z_1, z_2) = 0$ in \mathbb{P}^2 .

Denote by $P'(z_0, z_2)$ and $Q'(z_1, z_2)$ the homogenization of the polynomials $P'(x)$ and $Q'(y)$ respectively. Hence

$$\begin{aligned}\frac{\partial F}{\partial z_0} &= P'(z_0, z_2) = na_n(z_0 - \alpha_1 z_2)^{p_1} \dots (z_0 - \alpha_l z_2)^{p_l}, \\ \frac{\partial F}{\partial z_1} &= z_2^{n-m} Q'(z_1, z_2) = mb_m z_2^{n-m} (z_1 - \beta_1 z_2)^{q_1} \dots (z_1 - \beta_h z_2)^{q_h}, \\ \frac{\partial F}{\partial z_2} &= z_2^{n-m'-1} [s z_2^{m'-n_0} z_0^{n_0} + t z_2^{m'-m''} z_1^{m''} + z_2 E(z_0, z_1, z_2)]\end{aligned}$$

where s and t are constants such that $st \neq 0$, $E(z_0, z_1, z_2)$ is a homogeneous polynomial of degree $m' - 1$ which can be calculated explicitly and depend only on P and Q , and

$$\begin{aligned}m' &= \max\{n_0, m_0\} \text{ if } n = m \text{ and } m' = \max\{n_0, m\} \text{ if } n > m, \\ m'' &= m_0 \text{ if } n = m \text{ and } m'' = m \text{ if } n > m.\end{aligned}$$

Lemma 8 ([1, Lemma 4]). *The only possible singular points of the projective curve C are $(0 : 1 : 0)$ and the $(\alpha_i : \beta_j : 1)$ such that $P(\alpha_i) = Q(\beta_j)$, for $1 \leq i \leq l$ and $1 \leq j \leq h$. Moreover, if $n = m$ then the curve has no singularity at infinity.*

Proof. Suppose $a = (a_1, a_2, a_3)$ is a singularity, hence

$$\frac{\partial F}{\partial z_0}(a) = \frac{\partial F}{\partial z_1}(a) = \frac{\partial F}{\partial z_2}(a) = 0.$$

If $a_3 = 1$ then $\frac{\partial F}{\partial z_0}(a) = \frac{\partial F}{\partial z_1}(a) = 0$ and $P(a_1) = Q(a_2)$. Hence $a_1 = \alpha_i$ and $a_2 = \beta_j$ and $P(\alpha_i) = Q(\beta_j)$, for $1 \leq i \leq l$ and $1 \leq j \leq h$.

If $a_3 = 0$ then $\frac{\partial F}{\partial z_0}(a) = na_1^{n-1} = 0$ hence $a_1 = 0$.

Now, if $a = (a_1, a_2, 0)$ is a singularity at infinity and $n = m$, then $\frac{\partial F}{\partial z_0}(a) = na_n a_1^{n-1} = 0$ and $\frac{\partial F}{\partial z_1}(a) = nb_n a_2^{n-1} = 0$ hence $a_1 = a_2 = 0$, which is impossible. Therefore the curve has no singularity at infinity when $n = m$. \square

3.1. Proof of Theorem 1. In Theorem 1 we consider $P(x)$ and $Q(x)$ to be polynomials of the same degrees. Hence

$$\begin{aligned}\frac{\partial F}{\partial z_0} &= P'(z_0, z_2) = na_n(z_0 - \alpha_1 z_2)^{p_1} \dots (z_0 - \alpha_l z_2)^{p_l}, \\ \frac{\partial F}{\partial z_1} &= Q'(z_1, z_2) = mb_m(z_1 - \beta_1 z_2)^{q_1} \dots (z_1 - \beta_h z_2)^{q_h}, \\ \frac{\partial F}{\partial z_2} &= z_2^{n-m'-1} [s z_2^{m'-n_0} z_0^{n_0} + t z_2^{m'-m_0} z_1^{m_0} + z_2 E(z_0, z_1, z_2)]\end{aligned}$$

where $m' = \max\{n_0, m_0\}$.

Proof of Theorem 1. Consider

$$\omega_1 := \frac{W(z_0, z_1)}{z_2^2}, \quad \text{and } \omega_2 := \frac{z_0 W(z_0, z_1)}{z_2^3}.$$

They are well-defined rational 1-forms of Wronskian type and have a possible pole at infinity (i.e at $z_2 = 0$). By Lemma 8, when $m = n$ the curve has no singularity at infinity. It follows that ω_1 and ω_2 are regular at every singular point. It is easy to see from the hypothesis that $P(x) - Q(y)$ has no linear factor and that ω_1 and ω_2 are \mathbb{C} -linearly independent on any irreducible component of the curve C . However, by the hypothesis $n \geq m' := \max\{n_0, m_0\} + 4$, their denominators are factors of $z_2^{n-m'-1}$ and hence also of $\frac{\partial F}{\partial z_2}$. Therefore by the Key Lemma, every irreducible component of the curve C has genus at least 2, and hence the equation $P(f) = Q(g)$ has no non-constant meromorphic function solutions. \square

3.2. Proof of Theorem 2. From here, we always assume that the polynomials P and Q satisfy hypothesis I.

In the proof of Theorem 2, we will need the following lemmas. First, when the polynomials P and Q satisfy hypothesis I, we will give an upper bound on the cardinality of A_0 .

Lemma 9. *Let $P(X)$ and $Q(X)$ satisfy Hypothesis I. Then for each i , $1 \leq i \leq l$, there exists at most one j , $1 \leq j \leq h$, such that $P(\alpha_i) = Q(\beta_j)$. Moreover, $l_0 \leq \min\{l, h\}$.*

Proof. For each i , ($1 \leq i \leq l$), assume that there exist j_1, j_2 , $1 \leq j_1, j_2 \leq h$, such that $P(\alpha_i) = Q(\beta_{j_1})$ and $P(\alpha_i) = Q(\beta_{j_2})$. This implies that $Q(\beta_{j_1}) = Q(\beta_{j_2})$ and hence $j_1 = j_2$ because Q satisfies Hypothesis I. Similarly, there exists at most one i , ($1 \leq i \leq l$) such that $P(\alpha_i) = Q(\beta_j)$ for each j , ($1 \leq j \leq h$). This ends the proof of Lemma 9. \square

Recall that we have set:

$$\begin{aligned} A_0 &:= \{(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq h, P(\alpha_i) = Q(\beta_j)\}, \\ A_1 &:= \{(i, j) \mid (i, j) \in A_0, p_i > q_j\}, \\ A_2 &:= \{(i, j) \mid (i, j) \in A_0, p_i < q_j\}, \end{aligned}$$

and we put $l_0 := \#A_0$.

By Lemma 9, without loss of generality we may assume that

$$\begin{aligned} A_0 &= \{(1, \tau(1)), \dots, (l_0, \tau(l_0))\}; \\ A_1 &= \{(1, \tau(1)), \dots, (l_1, \tau(l_1))\}, \end{aligned}$$

which we do from now on. In what follows, let $L_{i,j}$, $1 \leq i \neq j \leq l_0$, be the linear form associated to the line passing through the two points $(\alpha_i, \beta_{\tau(i)}, 1)$ and $(\alpha_j, \beta_{\tau(j)}, 1)$. Note that $L_{i,j}$ is defined by

$$\begin{aligned} L_{i,j} &:= (z_1 - \beta_{\tau(j)}z_2) - \frac{\beta_{\tau(i)} - \beta_{\tau(j)}}{\alpha_i - \alpha_j}(z_0 - \alpha_jz_2) \\ &= (z_1 - \beta_{\tau(i)}z_2) - \frac{\beta_{\tau(i)} - \beta_{\tau(j)}}{\alpha_i - \alpha_j}(z_0 - \alpha_i z_1). \end{aligned}$$

Lemma 10. *Let $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1) \in C$, $i = 1, \dots, l_0$.*

- (i) *Assume that $L_{i,j}$, $1 \leq i \neq j \leq l_0$, is not identically zero on any component of C . Then,*

$$\text{ord}_{\mathbf{p}_i} L_{i,j} \geq \min\{\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2), \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2)\},$$

for each local parameterization at \mathbf{p}_i and for each local parameterization at \mathbf{p}_j

$$\text{ord}_{\mathbf{p}_j} L_{i,j} \geq \min\{\text{ord}_{\mathbf{p}_j}(z_0 - \alpha_j z_2), \text{ord}_{\mathbf{p}_j}(z_1 - \beta_{\tau(j)} z_2)\}.$$

- (ii) $(p_i + 1) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) = (q_{\tau(i)} + 1) \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2)$.
 (iii) $\text{ord}_{\mathbf{p}_i} W(z_1, z_2) \geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2) - 1$.

Proof. (i) follows directly from the definition of $L_{i,j}$.

- (ii) By the following expansion of $P(x)$ and $Q(x)$:

$$P(x) = P(\alpha_i) + \sum_{j=p_i+1}^n \nu_{i,j}(x - \alpha_i)^j, \quad \text{and} \quad Q(x) = Q(\beta_{\tau(i)}) + \sum_{j=q_i+1}^m \mu_{i,j}(x - \beta_{\tau(i)})^j$$

where $\nu_{i,p_i+1}, \nu_{i,n}, \mu_{i,q_{\tau(i)}+1}$ and $\mu_{i,m}$ are non-zero constants. If $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1) \in C$, then $F(z_0, z_1, z_2)$ can be expressed in terms of $z_0 - \alpha_i z_2$ and $z_1 - \beta_{\tau(i)} z_2$ as

$$\begin{aligned} F(z_0, z_1, z_2) &= \nu_{i,p_i+1}(z_0 - \alpha_i z_2)^{p_i+1} + \{\text{terms in } z_0 - \alpha_i z_2 \text{ of higher degrees}\} \\ &\quad + \mu_{i,q_{\tau(i)}+1}(z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}+1} + \{\text{terms in } z_1 - \beta_{\tau(i)} z_2 \text{ of higher degrees}\}. \end{aligned}$$

The lemma is proved by comparing the orders of the term of lowest degree and terms of higher degrees.

- (iii) As $z_2 \equiv 1$ on a neighborhood of \mathbf{p}_i ,

$$\text{ord}_{\mathbf{p}_i} W(z_1, z_2) = \text{ord}_{\mathbf{p}_i} dz_2 \geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2) - 1.$$

□

Lemma 11. *The following assertions hold, on the curve C :*

- (i) *Given $i \in \{l_0 + 1, \dots, l\}$, $\eta_1 := \frac{W(z_1, z_2)}{(z_0 - \alpha_i z_2)^{p_i}}$ is regular at finite points.*
 (ii) *Given $i \in \{1, 2, \dots, l_0\}$. Then $\eta_2 := \frac{(z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} W(z_1, z_2)}{(z_0 - \alpha_i z_2)^{p_i}}$ is regular at finite points.*

- (iii) If $|p_i - q_{\tau(i)}| \leq 2$, then $\eta_3 := \frac{(z_1 - \beta_{\tau(i)} z_2) W(z_1, z_2)}{(z_0 - \alpha_i z_2)}$ is regular at \mathbf{p}_i , except when $p_i = 1$ and $q_{\tau(i)} = 3$.
- (iv) Given $i, j \in \{1, 2, \dots, l_0\}$ and integers u, v , let
- $$\zeta_{u,v} := \frac{L_{i,j}^u W(z_1, z_2)}{(z_0 - \alpha_i z_2)^v}.$$

Then

- (a) $\zeta_{u,v}$ is regular at $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1)$ if $|p_i - q_{\tau(i)}| \leq 1$, $u \geq v$ and $p_i \geq v$.
 Moreover, $\zeta_{2,1}$ is regular at $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1)$ if $|p_i - q_{\tau(i)}| \leq 2$.
- (b) $\zeta_{1,2}$ and $\zeta_{2,3}$ are regular at $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1)$ if $p_i = q_{\tau(i)} + 1$.

Proof. (i) At finite points, η_1 has possible pole when $z_2 = 1$ and $z_0 = \alpha_i$. However, by (4)

$$\begin{aligned} \eta_1 &= \frac{\prod_{j=1, \dots, l, j \neq i} (z_0 - \alpha_j z_2)^{p_j} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1} \dots (z_0 - \alpha_l z_2)^{p_l}} \\ &= \frac{\prod_{j=1, \dots, l, j \neq i} (z_0 - \alpha_j z_2)^{p_j} W(z_2, z_0)}{(z_0 - \beta_1 z_2)^{q_1} \dots (z_0 - \beta_h z_2)^{q_h}}, \end{aligned}$$

hence, $(\alpha_i, a, 1)$ is a pole if $a = \beta_k$ for some $k = 1, \dots, h$. By the definition of the set A_0 and l_0 , there does not exist any such a . We are done for (i).

(ii) Similar to the case (i), at finite points, η_2 has possible pole when $z_2 = 1$, $z_0 = \alpha_i$ and

$$\begin{aligned} \eta_2 &= \frac{(z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} \prod_{j=1, \dots, l, j \neq i} (z_0 - \alpha_j z_2)^{p_j} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1} \dots (z_0 - \alpha_l z_2)^{p_l}} \\ &= \frac{\prod_{j=1, \dots, l, j \neq i} (z_0 - \alpha_j z_2)^{p_j} W(z_2, z_0)}{\prod_{j=1, \dots, h, j \neq \tau(i)} (z_0 - \beta_j z_2)^{q_j}}. \end{aligned}$$

By the definition of the set A_0 , if $(\alpha_i, a, 1)$ is a pole then $a = \beta_{\tau(i)}$, but the term $(z_1 - \beta_{\tau(i)} z_2)$ is canceled in the denominator in the second part of the above formula. We are done for (ii).

(iii) From Lemma 10(ii), we have

$$(6) \quad (p_i + 1) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) = (q_{\tau(i)} + 1) \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2).$$

We first prove the following claim.

Claim. Assume that $b = \gcd(p_i + 1, q_{\tau(i)} + 1)$. Then $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq \frac{q_{\tau(i)} + 1}{b}$.

Furthermore, if $q_{\tau(i)} = p_i + 2$ and $p_i \geq 2$ then $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq \max\{3, \frac{p_i + 3}{2}\}$.

Indeed, we can write $p_i + 1 = bh_1$ and $q_{\tau(i)} + 1 = bh_2$, where h_1 and h_2 are relatively prime. From (6) we have $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq \frac{q_{\tau(i)} + 1}{b}$. If $q_{\tau(i)} = p_i + 2$ then $b = 1$ or $b = 2$. It follows that $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq \frac{q_{\tau(i)} + 1}{b} \geq \max\{3, \frac{p_i + 3}{2}\}$ if $p_i \geq 2$.

We now go back to prove the lemma.

If $p_i \geq q_{\tau(i)}$ then the lemma obviously holds because from (6) we have

$$\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \leq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2).$$

So, we may assume that $p_i < q_{\tau(1)} \leq p_i + 2$.

If $q_{\tau(i)} = p_i + 1$ then, by the above claim, $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq q_{\tau(i)} + 1 = p_i + 2$.

Hence

$$\begin{aligned} \text{ord}_{\mathbf{p}_i} \eta_3 &\geq \text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(i)} z_2) + \text{ord}_{\mathbf{p}_1} W(z_1, z_2) - \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_i z_2) \\ &\geq 2\text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(i)} z_2) - \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_i z_2) - 1 \\ &\geq \left(\frac{2(p_i + 1)}{q_{\tau(i)} + 1} - 1 \right) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \\ &\geq \frac{p_i}{p_i + 2} \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \geq 0. \end{aligned}$$

If $q_{\tau(i)} = p_i + 2$ and $p_1 \geq 2$ then by the claim, $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_1 z_2) \geq 3$. Hence

$$\begin{aligned} \text{ord}_{\mathbf{p}_i} \eta_3 &\geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2) + \text{ord}_{\mathbf{p}_i} W(z_1, z_2) - \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \\ &\geq 2\text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2) - \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \\ &\geq \left(\frac{2(p_i + 1)}{q_{\tau(i)} + 1} - 1 \right) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \\ &\geq \frac{3(p_i - 1)}{p_i + 3} - 1 \geq 0 \end{aligned}$$

if $p_i \geq 3$. If $p_i = 2$, then, by the claim, $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq \frac{5}{\gcd(3,5)} = 5$, hence $\text{ord}_{\mathbf{p}_i} \eta_3 \geq 0$. The assertion (iii) is proved.

(iv) If $p_i \leq q_{\tau(i)}$ then $\text{ord}_{\mathbf{p}_i} L_{i,j} \geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(i)} z_2) = \frac{(p_i + 1)}{q_{\tau(i)} + 1} \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_1 z_2)$.

Therefore,

$$\begin{aligned} \zeta_{u,v} &\geq (u + 1)\text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(1)} z_2) - v \cdot \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_2 z_2) - 1 \\ &\geq \left(\frac{(u + 1)(p_i + 1)}{q_{\tau(i)} + 1} - v \right) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_1 z_2) - 1. \end{aligned}$$

If $q_{\tau(i)} = p_i + 1$ then, by the above claim, $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq p_i + 2$. Hence, $\text{ord}_{\mathbf{p}_i} \zeta_{2,1} \geq 2p_i \geq 0$, and

$$\text{ord}_{\mathbf{p}_i} \zeta_{u,v} \geq (u - v)(p_i + 1) + (p_i - v) \geq 0 \text{ if } u \geq v \text{ and } p_i \geq v.$$

If $q_{\tau(i)} = p_i + 2$ then $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_2 z_2) \geq 2$ and we only consider for $\zeta_{2,1}$. We have

$$\text{ord}_{\mathbf{p}_i} \zeta_{2,1} \geq \frac{4p_i}{p_i + 3} - 1 = \frac{3p_i - 3}{p_i + 3} \geq 0$$

for every $p_i \geq 1$.

If $p_i > q_{\tau(i)}$, then $\text{ord}_{\mathbf{p}_i} L_{i,j} \geq \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2)$, hence the assertions (a) obviously holds. For the assertion (b), we have $p_i = q_{\tau(i)} + 1$. Therefore, by the claim, we

have $\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \geq p_i$ and hence

$$\begin{aligned} \text{ord}_{\mathbf{p}_i} \zeta_{1,2} &\geq \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) + \text{ord}_{\mathbf{p}} W(z_1, z_2) - 2\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \\ &\geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(1)} z_2) - \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \\ &\geq \left(\frac{p_i + 1}{p_i} - 1\right) \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \geq 0, \end{aligned}$$

and

$$\begin{aligned} \text{ord}_{\mathbf{p}_i} \zeta_{2,3} &\geq 2\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) + \text{ord}_{\mathbf{p}} W(z_1, z_2) - 3\text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) \\ &\geq \text{ord}_{\mathbf{p}_i}(z_1 - \beta_{\tau(1)} z_2) - \text{ord}_{\mathbf{p}_i}(z_0 - \alpha_i z_2) - 1 \geq 0. \end{aligned}$$

Thus, the lemma 11 is proved. \square

Proof of Theorem 2. Consider

$$\begin{aligned} \omega_1 &:= \frac{z_0 z_2^{\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 3} \prod_{i=1}^{l_1} (z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} W(z_1, z_2)}{\prod_{i=1}^{l_1} (z_0 - \alpha_i z_2)^{p_i} \prod_{i=l_0+1}^l (z_0 - \alpha_i z_2)^{p_i}}, \\ \omega_2 &:= \frac{z_2^{\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 2} \prod_{i=1}^{l_1} (z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} W(z_1, z_2)}{\prod_{i=1}^{l_1} (z_0 - \alpha_i z_2)^{p_i} \prod_{i=l_0+1}^l (z_0 - \alpha_i z_2)^{p_i}}. \end{aligned}$$

They are well-defined rational 1-forms of Wronskian type and are \mathbb{C} -linearly independent on any irreducible component of the curve C because of the hypothesis that $P(x) - Q(y)$ has no linear factor. However, by the hypothesis

$$\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 3 \geq n - m \geq 0,$$

their denominators are factors of $\frac{\partial F}{\partial z_0}$. We will prove they are regular at every singular point of the curve C . By Lemma 11(ii), ω_1, ω_2 are regular at $(\alpha_i, \beta_{\tau(i)}, 1)$ for $i = 1, \dots, l_1$. By Lemma 11(i), ω_1, ω_2 are regular at finite singular points for any $j = l_0 + 1, \dots, l$. We only have to check the regularity of ω_1, ω_2 at $(0, 1, 0)$. By (4) and the fact

$$\frac{\partial F}{\partial z_1} = mb_m z_2^{n-m} (z_1 - \beta_1 z_2)^{q_1} \dots (z_1 - \beta_h z_2)^{q_h},$$

we have

$$\begin{aligned} \omega_1 &= \frac{z_0 z_2^{\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 3} \prod_{i=1}^{l_1} (z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} \prod_{i=l_1+1}^{l_0} (z_0 - \alpha_i z_2)^{p_i} W(z_1, z_2)}{\prod_{i=1}^l (z_0 - \alpha_i z_2)^{p_i}} \\ &= \frac{z_0 z_2^{\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 3} \prod_{i=1}^{l_1} (z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} \prod_{i=l_1+1}^{l_0} (z_0 - \alpha_i z_2)^{p_i} W(z_2, z_0)}{mb_m z_2^{n-m} (z_1 - \beta_1 z_2)^{q_1} \dots (z_1 - \beta_h z_2)^{q_h}} \\ &= \frac{z_0 z_2^{\sum_{i=1}^{l_1} (p_i - q_{\tau(i)}) + \sum_{i=l_0+1}^l p_i - 3 - n + m} \prod_{i=1}^{l_1} (z_1 - \beta_{\tau(i)} z_2)^{q_{\tau(i)}} \prod_{i=l_1+1}^{l_0} (z_0 - \alpha_i z_2)^{p_i} W(z_2, z_0)}{mb_m (z_1 - \beta_1 z_2)^{q_1} \dots (z_1 - \beta_h z_2)^{q_h}}, \end{aligned}$$

which implies $(0, 1, 0)$ can not be a pole of ω_1 . \square

3.3. Proof of theorem 4. In this section, we always assume $n = m$.

Lemma 12. *Assume that the curve C has no linear component. If one of the following holds then the curve C cannot have an irreducible component of genus 0 or 1.*

- (a) $l_0 \geq 2$ and $\sum_{i=1}^{l_1} (p_i - q_i) + \sum_{i=l_0+1}^l p_i = 2$,
- (b) $l_0 \geq 1$ and $\sum_{i=l_0+1}^l p_i = 2$, except the case when $l_0 = 1$ and $p_1 = 1, q_{\tau(1)} = 3$.
- (c) $l_0 \geq 2$ and $l = l_0 + 1$, except when $l_0 = 2, p_{l_0+1} = 1$ and $p_1 = p_2 = 1$.

Proof. By possibly rearranging the indices, we only have to consider the following cases:

- (1) $l_0 \geq 2$ and $(1, \tau(1)), (2, \tau(2)) \in A_0$ such that $p_1 - q_{\tau(1)} = 2$.
- (2) $l_0 \geq 2$ and $(1, \tau(1)), (2, \tau(2)) \in A_1$ such that $p_i - q_{\tau(i)} = 1$ with $i = 1, 2$.
- (3) $l_0 \geq 2$ and $(1, \tau(1)) \in A_1$ such that $p_1 - q_{\tau(1)} = 1$ and $l = l_0 + 1$ and $|p_j - q_{\tau(j)}| \leq 1$ for every $j = 1, \dots, l_0$.
- (4) $l_0 \geq 1$ and $\sum_{i=l_0+1}^l p_i = 2$, except when $l_0 = 1$ and $p_1 = 1, q_{\tau(1)} = 3$.
- (5) $l_0 \geq 2$ and $l = l_0 + 1$ and $p_{l_0+1} = 1$ and $0 \leq q_{\tau(i)} - p_i \leq 1$ with $i = 1, 2, \dots, l_0$, except when $l_0 = 2, p_{l_0+1} = 1$ and $p_1 = p_2 = 1$.

(Note that in the case 3, if $|p_j - q_{\tau(j)}| \geq 3$ then we are done because of Theorem 2, if there exists j , ($j \in \{1, \dots, l_0\}$), such that $p_j - q_{\tau(j)} = 2$ then we go back to the case 1, if $q_{\tau(j)} - p_j = 2$ then proceed as in case 1. Therefore, we could assume $|p_j - q_{\tau(j)}| \leq 1$ for any $j = 1, \dots, l_0$.)

Corresponding to each case, we will construct two rational 1-forms of Wronskian type which satisfy all the conditions of the Key lemma.

(1)

$$\begin{aligned}\omega_{1,1} &= \frac{(z_1 - \beta_{\tau(1)} z_2)^{p_1-2} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1}}, \\ \omega_{1,2} &= \frac{L_{1,2}^2 (z_1 - \beta_{\tau(1)} z_2)^{p_1-3} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1} (z_0 - \alpha_2 z_2)}.\end{aligned}$$

(2)

$$\begin{aligned}\omega_{2,1} &= \frac{(z_1 - \beta_{\tau(1)} z_2)^{p_1-1} (z_1 - \beta_{\tau(2)} z_2)^{p_2-1} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1} (z_0 - \alpha_2 z_2)^{p_2}}, \\ \omega_{2,2} &= \frac{L_{1,2} (z_1 - \beta_{\tau(1)} z_2)^{p_1-2} (z_1 - \beta_{\tau(2)} z_2)^{p_2-1} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^{p_1} (z_0 - \alpha_2 z_2)^{p_2}}.\end{aligned}$$

(3)

$$\begin{aligned}\omega_{3,1} &= \frac{L_{1,2}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2 (z_0 - \alpha_2 z_2) (z_0 - \alpha_{l_0+1} z_2)}, \\ \omega_{3,2} &= \frac{(z_1 - \beta_{\tau(1)} z_2) L_{12} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2 (z_0 - \alpha_2 z_2) (z_0 - \alpha_{l_0+1} z_2)}.\end{aligned}$$

(4)

$$\begin{aligned}\omega_{4,1} &= \frac{W(z_1, z_2)}{\prod_{i=l_0+1}^l (z_0 - \alpha_i z_2)}, \\ \omega_{4,2} &= \begin{cases} \frac{L_{1,2}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2) \prod_{i=l_0+1}^l (z_0 - \alpha_i z_2)} & \text{if } l_0 \geq 2 \\ \frac{(z_1 - \beta_{\tau(1)} z_2) W(z_1, z_2)}{(z_0 - \alpha_1 z_2) \prod_{i=l_0+1}^l (z_0 - \alpha_i z_2)} & \text{Otherwise, except when} \\ & p_1 = 1, \text{ and } q_{\tau(1)} = 3. \end{cases}\end{aligned}$$

(5) Assume $p_1 \geq p_2 \geq \dots \geq p_{l_0}$. Take

$$\omega_{5,1} = \frac{L_{1,2} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(z_0 - \alpha_{l_0+1} z_2)}$$

and

$$\omega_{5,2} = \begin{cases} \frac{L_{1,2}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2 (z_0 - \alpha_2 z_2) (z_0 - \alpha_{l_0+1} z_2)} & \text{if } p_1 \geq 2 \\ \frac{L_{1,2} L_{1,3} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2)(z_0 - \alpha_{l_0+1} z_2)} & \text{if } p_1 = 1 \text{ and } l_0 \geq 3. \end{cases}$$

We will show that the $\omega_{i,j}$'s satisfy condition (iii) in the Key lemma for all $i = 1, 2, \dots, 5$ and $j = 1, 2$. By Lemma 8, the curve C does not have any singular point at infinity, so we only have to prove they are regular at every point $\mathbf{p}_i = (\alpha_i, \beta_{\tau(i)}, 1)$, ($i = 1, \dots, l_0$) which are zeros of their respective denominators.

We now prove that the $\omega_{i,j}$'s are regular at $\mathbf{p}_1 = (\alpha_1, \beta_{\tau(1)}, 1)$. By Lemma 10,

$$\begin{aligned}(p_1 + 1) \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_1 z_2) &= (q_{\tau(1)} + 1) \text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(1)} z_2); \\ \text{ord}_{\mathbf{p}_1} W(z_1, z_2) &\geq \text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(1)} z_2) - 1;\end{aligned}$$

and if $p_1 \geq q_{\tau(1)}$ then $\text{ord}_{\mathbf{p}_1} L_{1,2} \geq \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_1 z_2)$. Hence,

$$\begin{aligned}\text{ord}_{\mathbf{p}_1} \omega_{1,2} &= \text{ord}_{\mathbf{p}_1} L_{1,2}^2 + \text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(1)} z_2)^{p_1-3} + \text{ord}_{\mathbf{p}_1} W(z_1, z_2) - \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_1 z_2)^{p_1} \\ &\geq (p_1 - 2) \text{ord}_{\mathbf{p}_1}(z_1 - \beta_{\tau(1)} z_2) - (p_1 - 2) \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_1 z_2) - 1 \\ &\geq (p_1 - 2) \left(\frac{p_1 + 1}{p_1 - 1} - 1 \right) \text{ord}_{\mathbf{p}_1}(z_0 - \alpha_1 z_2) - 1 \geq 0,\end{aligned}$$

because $\text{ord}_{\mathfrak{p}_1}(z_0 - \alpha_1 z_2) \geq 1$ and $p_1 - q_{\tau(1)} = 2$ so $p_1 \geq 3$. Therefore, $\omega_{1,2}$ is regular at \mathfrak{p}_1 .

$$\begin{aligned} \text{ord}_{\mathfrak{p}_1} \omega_{2,2} &= \text{ord}_{\mathfrak{p}_1} L_{1,2} + \text{ord}_{\mathfrak{p}_1} (z_1 - \beta_{\tau(1)} z_2)^{p_1-2} + \text{ord}_{\mathfrak{p}_1} W(z_1, z_2) - \text{ord}_{\mathfrak{p}_1} (z_0 - \alpha_1 z_2)^{p_1} \\ &\geq (p_1 - 1) \left(\frac{p_1 + 1}{p_1} - 1 \right) \text{ord}_{\mathfrak{p}_1} (z_0 - \alpha_1 z_2) - 1 \geq 0, \end{aligned}$$

because $\text{ord}_{\mathfrak{p}_1}(z_0 - \alpha_1 z_2) \geq 1$ and $p_1 - q_{\tau(1)} = 1$ so $p_1 \geq 2$. Therefore, $\omega_{2,2}$ is regular at \mathfrak{p}_1 .

Now, $\omega_{1,1}, \omega_{2,1}$ are regular at \mathfrak{p}_1 by Lemma 11(i).

We know $\omega_{3,1}, \omega_{3,2}$ are regular at \mathfrak{p}_1 because $p_1 > q_{\tau(1)}$, and so

$$\text{ord}_{\mathfrak{p}_1}(z_0 - \alpha_1 z_2) < \text{ord}_{\mathfrak{p}_1}(z_1 - \beta_{\tau(1)} z_2)$$

and $\text{ord}_{\mathfrak{p}_1}(z_0 - \alpha_1 z_2) < \text{ord}_{\mathfrak{p}_1} L_{1,2}$.

We know $\omega_{4,2}$ is regular at \mathfrak{p}_1 by Lemma 11(iv,a) if $l_0 \geq 2$ and by Lemma 11(iii) otherwise, except when $l_0 = 1$, $p_1 = 1$ and $q_{\tau(1)} = 3$.

We know $\omega_{5,1}$ and $\omega_{5,2}$ are regular at \mathfrak{p}_1 by Lemma 11(iv,a).

We know $\omega_{2,1}, \omega_{2,2}$ are regular at \mathfrak{p}_2 by Lemma 11 (i).

We know $\omega_{1,2}, \omega_{3,1}, \omega_{3,2}, \omega_{4,2}, \omega_{5,1}$ and $\omega_{5,2}$ are regular at \mathfrak{p}_2 by Lemma 11(iv,a).

By Lemma 11 (ii), $\omega_{i,j}$'s, with $i = 3, 4, 5$ and $j = 1, 2$, have no pole on $z_0 - \alpha_{l_0+1} = 0$; $\omega_{4,2}$ has no pole on $z_0 - \alpha_{l_0+2} = 0$.

We are therefore done with showing that the $\omega_{i,j}$'s satisfy condition (iii) in the Key lemma.

By the conditions on p_i , condition (i) in the Key lemma is also satisfied.

Because of the hypothesis that the curve C does not have any linear irreducible components, we have that $\omega_{i,1}$ and $\omega_{i,2}$ (with $i = 2, 3, 5$) are \mathbb{C} -linearly independent on any irreducible component of the curve C . On the other hand, we were able to construct $\omega_{1,1}$ and $\omega_{4,1}$ which are non-trivial regular 1-forms on any irreducible component of the curve C . Hence, every irreducible component has genus at least 1. If $\omega_{j,1}$ and $\omega_{j,2}$ (for $j = 1$, or 4) are \mathbb{C} -linearly dependent on some irreducible component of the curve C , then C must have a quadratic component, which contradicts the fact that any irreducible component has genus at least 1. Thus, condition (ii) in the Key lemma is satisfied. \square

Using similar arguments, we also get the following corollary.

Corollary 13. *Assume that the curve C has no linear component. If one of the following holds then the curve C is Brody hyperbolic:*

$$(a) \ l_0 \geq 2 \text{ and } \sum_{i=l_0+1}^{l_0} (q_{\tau(i)} - p_i) + \sum_{i=l_0+1}^h q_i = 2,$$

- (b) $l_0 \geq 1$ and $\sum_{i=l_0+1}^h q_i = 2$, except the case when $l_0 = 1$ and $q_{\tau(1)} = 1, p_1 = 3$.
(c) $l_0 \geq 2$ and $h = l_0 + 1$, except when $l_0 = 2, q_{l_0+1} = 1$ and $q_{\tau(1)} = q_{\tau(2)} = 1$.

Definition 14. Let $R(z_0, z_1, z_2) = 0$ be a curve of degree $\deg R$ over \mathbb{C} . Denote by δ_R the deficiency of the plane curve $R(z_0, z_1, z_2) = 0$ which is

$$\delta_R = \frac{1}{2}(\deg R - 1)(\deg R - 2) - \frac{1}{2} \sum_{\mathbf{p}} m_{\mathbf{p}}(m_{\mathbf{p}} - 1)$$

where the sum is taken over all points in $R(z_0, z_1, z_2) = 0$ and $m_{\mathbf{p}}$ is the multiplicity of $R(z_0, z_1, z_2) = 0$ at \mathbf{p} .

Proposition 15. Let \mathcal{C} be a curve in $\mathbb{P}^2(\mathbb{C})$ of degree n .

- (i) If \mathcal{C} has only one singular point and it is ordinary of multiplicity μ which is either $n - 1$ or $n - 2$, then \mathcal{C} is irreducible.
(ii) If \mathcal{C} has only two singular points and they are ordinary of multiplicity $n - 1$ and 2 respectively, then \mathcal{C} has a linear component.

Proof. Let \mathcal{C} be define by $F(z_0, z_1, z_2) = 0$ and let $H(z_0, z_1, z_2)$ be its proper irreducible factor of degree d . Then $F(z_0, z_1, z_2) = H(z_0, z_1, z_2)G(z_0, z_1, z_2)$ for some $G \in \mathbb{C}[z_0, z_1, z_2]$ and $1 \leq d < n$. Clearly, $G(z_0, z_1, z_2)$ is not divisible by $H(z_0, z_1, z_2)$ because $F(z_0, z_1, z_2) = 0$ has only finitely many singular points.

(i) Let m_H be the multiplicity of the singular point in $H(z_0, z_1, z_2) = 0$. By Bezout's theorem we have

$$(7) \quad d(n - d) = m_H(\mu - m_H).$$

Since the multiplicity of the point in the intersection of these two curves is not bigger than the degree of each curve, it follows that

$$m_H \leq d \text{ and } \mu - m_H \leq n - d.$$

Hence, $m_H \leq d \leq n - \mu + m_H$, where μ is either $n - 1$ or $n - 2$. This is impossible if $1 \leq d < n$. Hence, $F(z_0, z_1, z_2)$ is irreducible, and we are done for (i).

(ii) In this case, the curve has deficiency $\delta_{\mathcal{C}} = \frac{p_1(p_1 + 1)}{2} - \frac{p_1(p_1 + 1)}{2} - 1 < 0$. Therefore, the curve is reducible and using the above argument for the case $\mu = n - 1$, the curve $H(z_0, z_1, z_2) = 0$ has to pass through both of the singular points. By Bezout's theorem,

$$(n - d)d = m_H(n - 1 - m_H) + 1,$$

from which it follows that $d = 1$. Therefore the curve has a linear component, and we are done for (ii). \square

The following lemma is a special case of [2, proposition 6]. For the convenience of the readers, we will recall here a brief proof.

Lemma 16. *Let the curve $C = \{F(z_0, z_1, z_2) = 0\}$ have only one singular point, say $(\alpha_1, \beta_{\tau(1)}, 1)$ such that $p_1 = 3$ and $q_{\tau(1)} = 1$. Then the curve C is birational to a curve $R(z_0, z_1, z_2) = 0$ with only ordinary singularities. Furthermore,*

$$\delta_R = \delta_C - 1.$$

Proof. We first make a linear transformation which takes the curve to an excellent position, and the point $(\alpha_i, \beta_{t(i)}, 1)$ to the origin. Let

$$\begin{aligned} R_{01}(z_0, z_1, z_2) &= F(z_0 + \alpha_1 z_2, z_0 + z_1 + \beta_{t(1)} z_2, z_2) \\ &= \nu_1 z_0^4 z_2^{n-4} + \nu_2 z_0^5 z_2^{n-5} + \cdots + z_0^n \\ &\quad + \mu_1 (z_0 + z_1)^2 z_2^{n-2} + \mu_2 (z_0 + z_1)^3 z_2^{n-3} + \cdots - c(z_0 + z_1)^n \end{aligned}$$

where ν_i 's and μ_i 's are constant. We then perform a quadratic transformation

$$\begin{aligned} R_{01}(z_1 z_2, z_0 z_2, z_0 z_1) &= \nu_1 z_2^4 z_1^n z_0^{n-4} + \nu_2 z_2^5 z_1^n z_0^{n-5} + \cdots + z_2^n z_1^n \\ &\quad + \mu_1 \mu_1 z_2^2 (z_0 + z_1)^2 (z_0 z_1)^{n-2} + \mu_2 z_2^3 (z_0 + z_1)^3 (z_0 z_1)^{n-3} + \cdots - c z_2^n (z_0 + z_1)^n \\ &= z_2^2 R_1(z_0, z_1, z_2), \end{aligned}$$

where

$$\begin{aligned} R_1(z_0, z_1, z_2) &= \nu_1 z_2^2 z_1^n z_0^{n-4} + \nu_2 z_2^3 z_1^n z_0^{n-5} + \cdots + z_2^{n-2} z_1^n + \\ &\quad + \mu_1 (z_0 + z_1)^2 (z_0 z_1)^{n-2} + \mu_2 z_2 (z_0 + z_1)^3 (z_0 z_1)^{n-3} + \cdots - c z_2^{n-2} (z_0 + z_1)^n. \end{aligned}$$

For points of $F(z_0, z_1, z_2) = 0$ outside of the union of the 3 exceptional lines $\{z_0 = 0\}$, $\{z_1 = 0\}$, and $\{z_2 = 0\}$, these transformations preserve the multiplicities and ordinary multiple points. It is easy to see that the 3 fundamental points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ become ordinary multiple points of $R_1(z_0, z_1, z_2) = 0$ with multiplicities $n - m_i - 1$, $n - m_i - 1$, and n respectively. We also check that the only non-fundamental point in the intersection of $R_1(X, Y, Z)$ with the union of three exceptional lines is $\mathbf{q}_1 = (1, -1, 0)$. Since

$$R_1(1, z_1, z_2) = \nu_1 z_2^2 z_1^n + \cdots + z_2^{n-2} z_1^n + \mu_1 (1 + z_1)^2 z_1^{n-2} + \cdots - c z_2^{n-2} (1 + z_1)^n,$$

the point \mathbf{q}_1 is an ordinary multiple point of multiplicity 2 and $\delta_{R_1} = \delta_{F_c} - 1$. \square

Lemma 17. *Assume that P and Q are polynomials satisfying Hypothesis I and $\deg P = \deg Q$. If one of the following holds, then the curve C has an irreducible component of genus 0 or 1.*

- (1) $P(X) - Q(Y)$ has a linear factor.
- (2) $n = 2$ or $n = 3$.
- (3) $n = 4$ and either there exists at least two i such that $P(\alpha_i) = Q(\beta_{\tau(i)})$ or there exists only one i such that $P(\alpha_i) = Q(\beta_{\tau(i)})$ and $|p_i - q_{\tau(i)}| = 2$.
- (4) either $n = p_1 + 1, l = 1, h = 2, p_1 = q_1 + 1, q_2 = 1$ and $P(\alpha_1) = Q(\beta_1)$; or $n = p_1 + 2, h = 1, l = 2, q_1 = p_1 + 1, p_2 = 1$ and $P(\alpha_1) = Q(\beta_1)$.
- (5) $l = h = 2, p_2 = q_2 = 1, p_1 = q_1, n = p_1 + 2$, and $P(\alpha_1) = Q(\beta_1)$.
- (6) $n = 5, l_0 = l = h = 3, p_3 = p_2 = q_2 = q_3 = 1, p_1 = q_{\tau(1)} = 2, P(\alpha_i) = Q(\beta_{\tau(i)})$, for $i = 1, 2, 3$.
- (7) $n = 5, l_0 = l = h = 2, p_i = q_i = 2, P(\alpha_i) = Q(\beta_{\tau(i)})$, for $i = 1, 2$.

Proof. For cases (1) and (2), the curve clearly has a component of genus 0 or 1.

In case (3), because the curve C has degree $n = 4$, if it is reducible then it has either a linear component or a quadratic factor, which has genus 0. Assume that C is irreducible. If there exists at least two i such that $P(\alpha_i) = Q(\beta_{\tau(i)})$ then the curve has at least two singular points and its genus is at most $\frac{(4-1)(4-2)}{2} - 2 = 1$. If there exists only one i such that $P(\alpha_i) = Q(\beta_{\tau(i)})$ and $|p_i - q_{\tau(i)}| = 2$, then by Lemma 16, the curve is birational to a curve of genus $\delta_C - 1 = \frac{(4-1)(4-2)}{2} - 1 - 1 = 1$.

In case (4), the curve C of degree n has only one singular point $(\alpha_1, \beta_{\tau(1)}, 1)$ of multiplicity $n - 1$. Its deficiency $\delta_C = \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-1)(n-2) = 0$. On the other hand, locally near the singular point, one can write

$$P(z_0) - Q(z_1) = u_1(z_0 - \alpha_1)^n - v_1(y - \beta_{\tau(1)})^n - v_2(y - \beta_{\tau(1)})^{n-1}$$

which is easily seen to be irreducible. Therefore, the curve C has genus zero.

In case (5), by Proposition 15, the curve C is irreducible. Hence, we have the deficiency of the curve C is its genus which is $\mathfrak{g}_C = \delta_C = \frac{(n-1)(n-2)}{2} - \frac{(p_1+1)(p_1)}{2} = 0$.

For cases (6) and (7), because the curve C has degree $n = 5$, if it is reducible, then it has either a linear factor or a quadratic factor, which therefore has genus 0. Assume that C is irreducible. In case (6), the curve has 3 singular points which are all ordinary, so its genus is $\frac{(5-1)(5-2)}{2} - \frac{3(3-1)}{2} - \frac{2(2-1)}{2} - \frac{2(2-1)}{2} = 1$. In case (7), the curve has 2 singular points which are all ordinary of multiplicity 3. So its genus is $\frac{(5-1)(5-2)}{2} - 2 \cdot \frac{3(3-1)}{2} = 0$. \square

Proof of Theorem 4. By Lemma 17, if the polynomials P and Q satisfy one of the cases (1) ,..., (7), then the curve C has an irreducible component of genus 0 or 1. We now assume they do not fall into any of the cases (1) ,..., (7). Since $P(x)$ and $Q(x)$ are not linear polynomials, we can assume both l and h are not zero.

We will consider the following cases.

Case 1. $l_0 = 0$.

In this case, the curve C does not have any singular points. Therefore, it is irreducible of genus $g_C = \frac{1}{2}(n-1)(n-2) \geq 0$ if $n \geq 4$.

Case 2. $l_0 = 1$.

If $l_0 = 1$, and either $p_1 = 1, q_{\tau(1)} = 3$ and $\sum_{i=l_0+1}^l p_i = 2$, or $p_1 = 3, q_{\tau(1)} = 1$ and $\sum_{i=l_0+1}^l q_{\tau(i)} = 2$, then $n = 4$. This is the exceptional case 3. By Theorem 2 and Lemma 12 (b), we only have to consider when $l \leq l_0 + 1 = 2$ and $p_j = 1$ for all $j = 2, \dots, l$. Similarly, $h \leq 2$ and $q_i = 1$ for all $i \neq \tau(1)$. Therefore, the remaining cases are

$$(8) \quad \begin{aligned} &|p_1 - q_{\tau(1)}| \leq 2, \max(l, h) \leq l_0 + 1 = 2, q_i = 1 \text{ for all } i \neq \tau(1), \\ &\text{and } p_j = 1 \text{ for all } j = 2, \dots, l. \end{aligned}$$

We will consider the following sub-cases.

Subcase 1. $l = 1$ and $h = 1$.

In this case $P(x) - Q(y) = (x - \alpha_1)^n - (y - \beta_1)^n$ has linear factors, this is the exceptional case 1.

Subcase 2. $l = 1$ and $h = 2$ (or $l = 2$ and $h = 1$).

Since $n = m$, it follows that $p_1 = q_{\tau(1)} + q_{i, i \neq \tau(1)}$. By the condition (8), $q_i = 1$ for $i \neq \tau(1)$, we have $n - 1 = p_1 = q_{\tau(1)} + 1$. This is the exceptional case 4.

Similarly, $l = 2, h = 1, q_{\tau(1)} = p_1 + 1$ and $p_2 = 1$ is the exceptional case 4.

Subcase 3. $l = 2$ and $h = 2$.

By Theorem 2, we may assume that $|p_1 - q_{\tau(1)}| \leq 1$. However, by the assumption $n = m$, we have $n - 1 = p_1 + p_2 = q_{\tau(1)} + q_{i, i \neq \tau(1)}$. Since $p_2 = q_{i, i \neq \tau(1)} = 1$ by (8), we have $p_1 = q_{\tau(1)}$ and $n = p_1 + 2$. This is the exceptional case 5.

Case 3. $l_0 \geq 2$.

If $l_0 = 2, l = l_0 + 1 = 3$ and $p_{l_0+1} = 1$, and $p_1 = p_2 = 1$ then $n = 4$, and this is the exceptional case 3.

By Lemma 12, we only have to consider $|p_i - q_{\tau(i)}| \leq 1$, for every $i = 1, \dots, l_0$ and $l = l_0$, (and, similarly, $h = l_0$).

Without loss of generality, assume that $p_1 \geq p_2 \geq \dots \geq p_{l_0}$. Take

$$\omega = \frac{L_{12}^3 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^3 (z_0 - \alpha_2 z_2)^3}.$$

Hence, by Lemma 11(iv,a), ω is regular, from which it follows that

$$\omega_1 = z_0\omega, \quad \text{and} \quad \omega_2 = z_1\omega$$

are two regular well-defined 1-forms. Because the curve C does not have any linear components, they are \mathbb{C} -linearly independent on every irreducible component. However, if $p_2 \geq 3$ then the denominator of ω is a factor of $\frac{\partial F}{\partial z_0}$. So, all of the conditions in the Key Lemma are satisfied.

All together, the remaining cases are $l = h = l_0$, $|p_i - q_{\tau(i)}| \leq 1$, for every $i = 1, \dots, l_0$ and either $p_2 = 1$ or $p_2 = 2$. We have followings:

Subcase 1. $p_2 = 1$ and $l_0 = 2$.

Since $p_2 = 1$, we have $q_{\tau(2)} \leq p_2 + 1 = 2$.

If $q_{\tau(2)} = 1$ then, by $n = m$, $p_1 = q_{\tau(1)}$, and the curve has degree $p_1 + 2$ and has two singular points of multiplicity 2 and $p_1 + 1$, all of which are ordinary. By Proposition 15(ii), the curve has a linear factor, which is exceptional case 1.

If $q_{\tau(2)} = 2$, then $p_1 = q_{\tau(1)} + 1 \geq 2$. If $p_1 = 2$, then this is exceptional case 3. If $p_1 \geq 3$, then we consider

$$\begin{aligned} \gamma_{1,1} &= \frac{L_{12}W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2(z_0 - \alpha_2 z_2)}, \\ \gamma_{1,2} &= \frac{L_{12}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^3(z_0 - \alpha_2 z_2)}, \end{aligned}$$

which are regular 1-forms by Lemma 11, which are \mathbb{C} -linearly independent because the curve does not have any linear components, and are such that their denominators are factors of $\frac{\partial F}{\partial z_0}$ by $p_1 \geq 3$. So, all of the conditions in the Key Lemma are satisfied.

Subcase 2. $p_2 = 1$ and $l_0 = 3$.

If $p_1 = 1$, then $p_i = q_{\tau(i)} = 1$ for all $i = 1, 2, 3$ (because we assumed $p_1 \geq p_2 \geq \dots \geq p_{l_0}$). This is the exceptional case 3.

If $p_1 \geq 2$, then either $q_{\tau(1)} = p_1$, or $p_1 = q_{\tau(1)} + 1$ (by $l = h = l_0$). Consider two well-defined 1-forms

$$\begin{aligned} \gamma_{2,1} &= \frac{L_{12}L_{13}W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2(z_0 - \alpha_2 z_2)(x - \alpha_3 z_2)}, \\ \gamma_{2,2} &= \begin{cases} \frac{L_{12}L_{23}W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2(z_0 - \alpha_2 z_2)(x - \alpha_3 z_2)} & \text{if } p_1 = 2 \text{ and } q_1 = 1 \\ \frac{L_{12}L_{13}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^3(z_0 - \alpha_2 z_2)(x - \alpha_3 z_2)} & \text{if } p_1 \geq 3. \end{cases} \end{aligned}$$

They are regular by Lemma 11 and nontrivial on every irreducible component of the curve C because C does not have any linear components. However, if they are \mathbb{C} -linearly dependent, then the curve C has a quadric factor, which is impossible

since we can construct at least one regular nontrivial 1-form $\gamma_{2,1}$. The condition on p_i ensures that condition (i) of the Key Lemma is satisfied. Therefore, all the conditions in the Key Lemma are satisfied.

If $p_1 = q_{\tau(1)} = 2$, then it is exceptional case 6.

Subcase 3. $p_2 = 1, l_0 \geq 4$.

In this case, we consider

$$\gamma_{3,1} = \frac{L_{12}L_{34}W(z_1, z_2)}{(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(x - \alpha_3 z_2)(x - \alpha_4 z_2)},$$

$$\gamma_{3,1} = \frac{L_{13}L_{24}W(z_1, z_2)}{(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(x - \alpha_3 z_2)(x - \alpha_4 z_2)},$$

and it is easy to see they satisfy all the conditions in the Key Lemma.

Subcase 4. $p_2 = 2$.

In this case, $p_1 \geq 2$ and using same arguments as above, we can show the following 1-forms satisfy all the conditions in the Key Lemma:

$$\gamma_{4,1} = \frac{L_{12}^2 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2 (z_0 - \alpha_2 z_2)^2},$$

$$\gamma_{4,2} = \begin{cases} \frac{L_{12}^2 L_{13} W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^2 (z_0 - \alpha_2 z_2)^2 (z_0 - \alpha_3 z_2)} & \text{if } l_0 \geq 3 \\ \frac{L_{12}^3 W(z_1, z_2)}{(z_0 - \alpha_1 z_2)^3 (z_0 - \alpha_2 z_2)^2} & \text{if } l_0 = 2 \text{ and } p_1 \geq 3. \end{cases}$$

If $l_0 = 2$ and $p_1 = 2$, then $n = 5$ and either $q_{\tau(1)} = q_{\tau(2)} = 2$, or $q_{\tau(1)} = 3$ and $q_{\tau(2)} = 1$. When $q_{\tau(1)} = 3$ and $q_{\tau(2)} = 1$, we work similarly to the subcase 1 when $p_2 = 1, p_1 \geq 3$ and $q_{\tau(1)} = q_{\tau(2)} = 2$, which means we can construct two regular 1-forms $\gamma_{1,1}$ and $\gamma_{1,2}$. When $q_{\tau(1)} = q_{\tau(2)} = 2$, it is the exceptional case 7. \square

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